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UNIFORM THREE-CLASS REGULAR PARTIAL STEINER TRIPLE SYSTEMS WITH UNIFORM DEGREES

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Dedicated to Sri Sri Thakur who has given me strength and inspiration to lead this life…

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ABSTRACT

A Partial Steiner Triple system $(\mathcal{X}, \mathcal{T})$ is a finite set of points $\mathcal{X}$ and a collection $\mathcal{T}$ of 3-element subsets of $\mathcal{X}$ that every pair of points intersect in at most 1 triple. A 3-class regular PSTS (3-PSTS) written as 3- PSTS ($[m\cdot \alpha, n\cdot \beta, p\cdot \gamma]$) is a PSTS where the points can be partitioned into 3 classes (each class having size m, n and p respectively) such that no triple belongs to any class and any two points from the same class occur in the same number of triples (α, β and γ respectively). The 3-PSTS is said to be uniform if $m = n = p$. In this thesis, we have mostly focused on the existence of 3-PSTS ($[n\cdot \alpha, n\cdot \alpha, n\cdot \alpha]$).

In Chapter 1, we list some important definitions related to this topic along with examples and a brief look at the history and a construction of 3- PSTS ($[n\cdot n, n\cdot n, n\cdot n]$).

In Chapter 2, we obtain the results on the existence of 3-STS ($[n\cdot \alpha, n\cdot \alpha, n\cdot \alpha]$) and their constructions.

In Chapter 3, we provide constructions for the existence of 3- PSTS ($[n\cdot \alpha, n\cdot \alpha, n\cdot \alpha]$) when n is even based on the constructions from Chapter 2. We also determine for which values of n; we can achieve the maximum bound on α by using some known results on extended Skolem sequences.

In Chapter 4, we present a summary and some ideas for further research.

1. INTRODUCTION

A Steiner Triple System with parameters STS(v) is a pair $(\mathcal{X},\ \mathcal{T})$, where

1. $\mathcal{X}$ is a set of v *points*,
2. $\mathcal{T}$ is a collection of 3-element subsets of points, called *triples* and
3. Every pair of disjoint points occurs in exactly one triple.

1.1 Example: A Steiner triple system of order 9

$\mathcal{X} =\ \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$\mathcal{T}$ = {123, 456, 789, 147, 258, 369, 168, 249, 357, 159, 267, 348}

The following well-known result was proved by Kirkman in (1847).

Theorem 1.1 [1] *There exists an STS*(v) *if and only if* $v\ \equiv 1{,}3\ (mod\ 6), v\ \geq 7$.

1.2 Definitions

A Partial Steiner Triple System (PSTS) is a finite set of v points and a collection of 3-subsets (triples) where every pair of disjoint points occurs in at most one triple. A 3-class regular partial Steiner triple system $([m\cdot\alpha\ ,\ n\cdot\beta,\ p\cdot\gamma])$ is a triple $(\mathcal{X},\ [A,B,C],\ \mathcal{T})$ such that

- $(\mathcal{X},\ \mathcal{T})$ is a PSTS.
- $\mathcal{X}$ is partitioned into disjoint subsets A, B, C called the *classes,* such that $|A| = m$, $|B| = n, |C| = p$. Hence, $m + n + p = |\mathcal{X}| = v$.
- All the points in class A, B and C occur in α, β and γ triples respectively. Hence, defining $deg_{\mathcal{T}}(x)$ to be the number of triples in $\mathcal{T}$ that contain x, we require

$$deg_{\mathcal{T}}(x) = \begin{cases} \alpha\ if\ x\ \in A \\ \beta\ if\ x\ \in B \\ \gamma\ if\ x\ \in C \end{cases} \tag{1}$$

- No triple completely lies in any class.
- If $m = n = p$, then the 3-PSTS is called *uniform*.

Uniform 2-class regular PSTS were studied by Keranen, Kreher and Ozkan. They gave the following result in [2].

Theorem 1.2 [2] *A 2-PSTS* $(\{n\cdot\alpha, n\cdot\beta\})$ *exists with* $\alpha\ \geq\ \beta$ *if and only if*

1. $0\ \leq\ \left\lceil\frac{\alpha}{2}\right\rceil \leq\ \beta\ \leq\ \alpha\ \leq n-1$.
2. $n\ (\alpha +\ \beta) \equiv 0\ (mod\ 3)$.

3. $\alpha + \beta \le \frac{3n}{2}$
4. $n(2\alpha - \beta) \le 3\binom{n}{2}$.

In this thesis, we study uniform 3- class regular PSTS. We show the following results.

1. *There exists a 3- PSTS* $([n \cdot n, n \cdot n, n \cdot n])$ $\forall\ n \ge 1$.
2. *For every* $x \ge 1$, *there exists a uniform dense 3–STS* $([2x+1 \cdot 3x+1,\ 2x+1 \cdot 3x+1, 2x+1 \cdot 3x+1])$.
3. *There exists a 3-PSTS* $([2k \cdot (3k-2), 2k \cdot (3k-2), 2k \cdot (3k-2)])$ *for all positive integers* k.
4. *Let* $n = 2k$ *and* $k \equiv 0,3 \ (mod\ 12)$. *Then there exists a uniform dense 3-PSTS* $([2k \cdot (3k-1), 2k \cdot (3k-1), 2k \cdot (3k-1)])$.

1.3 3-class regular PSTS

In a 3- PSTS $([m \cdot \alpha, n \cdot \beta,\ p \cdot \gamma])$ $(\mathcal{X},\ [A, B, C],\ \mathcal{T})$, we denote a triple to be of Type (T)= ($|T \cap A|, |T \cap B|, |T \cap C|$). Since no triple can lie in any class, the triples are of Types $(210), (021), (120), (012), (201), (102)$ and (111).

1.3.1 Example- A uniform 3– PSTS$([4\ \cdot 1,\ 4\ \cdot 2,\ 4\ \cdot 3])$

$$A = \{\, a_1,\ a_2,\ a_3,\ a_4\}$$

$$B = \{\, b_1,\ b_2,\ b_3,\ b_4\}$$

$$C = \{\, c_1,\ c_2,\ c_3,\ c_4\}$$

$$\mathcal{T} = \begin{cases} \{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}, \{a_3, b_3, c_3\}, \\ \{b_0, c_2, c_3\}, \{b_1, c_3, c_0\}, \{b_2, c_0, c_1\}, \{b_3, c_1, c_2\} \end{cases}$$

Let N_{xyz} denote the number of triples of Type (xyz).

Here we can see that in this 3-PSTS there are 4 triples of Type (012) and 4 triples of Type (111).

Given a 3-PSTS $([m \cdot \alpha, n \cdot \beta,\ p \cdot \gamma])$ $(\mathcal{X},\ [A, B, C],\ \mathcal{T})$, we define the array $M : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \cup \{\emptyset\}$ by

$$M[a, b] = \begin{cases} c \text{ if } \{a, b, c\} \in \ \mathcal{T} \\ \emptyset \quad \text{otherwise} \end{cases} \tag{2}$$

The array $M : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \cup \{\emptyset\}$ corresponding to the 3-PSTS $([m \cdot \alpha, n \cdot \beta,\ p \cdot \gamma])$ $(\mathcal{X},\ [A, B, C],\ \mathcal{T})$ has the following properties:

1. M is semi-row Latin. The list $[M[x, y]: x, y \in A \cup B \cup C$, x $\neq y]$ contains each symbol in $A \cup B \cup C$ at most once.
2. M is semi-column Latin. Every $M[x, y]: x, y \in A \cup B \cup C, x \neq y$ contains each symbol in $A \cup B \cup C$ at most once.
3. Each entry on the main diagonal is $\emptyset$ since we cannot cover a pair (x, x) in any triple.
4. M is Steiner. This means $M[a, y] = z \in A \cup B \cup C, a \neq y$ iff $M\,[a, z] = y$.
5. M is row semi– regular of degree α, β and γ. This means the following conditions are satisfied.
   I. For each $a \in A, \frac{1}{2}(|\{M[a, x] \neq \emptyset,\ a \neq x, x \in A\}| + |\{M[a, y] \neq \emptyset,\ y \in B\}| + |\{M[a, z] \neq \emptyset,\ z \in C\}|) = \alpha$.
   II. For each $b \in B, \frac{1}{2}(|\{M[b, x] \neq \emptyset, x \in A\}| + |\{M[b, y] \neq \emptyset,\ b \neq y, y \in B\}| + |\{M[b, z] \neq \emptyset,\ z \in C\}|) = \beta$.
   III. For each $c \in C, \frac{1}{2}(|\{M[c, x] \neq \emptyset, x \in A\}| + |\{M[c, y] \neq \emptyset, y \in B\}| + |\{M[c, z] \neq \emptyset, c \neq z,\ z \in C\}|) = \gamma$.
6. Similarly, M is column semi – regular of degree α, β and γ.

We refer to the array $M : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \cup \{\emptyset\}$ fulfilling these six properties as a Steiner square $([m \cdot \alpha, n \cdot \beta, p \cdot \gamma])$. It can be easily verified that a Steiner square $([m \cdot \alpha, n \cdot \beta, p \cdot \gamma])$ is equivalent to a 3 – PSTS $([m \cdot \alpha, n \cdot \beta,\ p \cdot \gamma])$. For each row corresponding to a vertex a_i in class A, each triple containing a_i covers two entries in the same row. In other words, for a triple (a_i, x, y) we cover the cell (a_i, x) with y and the cell (a_i, y) with x. Since there are α triples containing a_i, 2α cells are filled in the row labelled as a_i. Similarly for each vertex in class B and class C, the number of filled cells in each row are 2β and 2γ respectively. Since there are m, n and p vertices in classes A, B and C respectively, the number of non-empty cells in a Steiner square $([m \cdot \alpha, n \cdot \beta, p \cdot \gamma])$ = $2\alpha m + 2\beta n + 2\gamma p = 2(\alpha m + \beta n + \gamma p)$.

1.4 Example 1.3.1 (Contd.)

The 3-PSTS $([4 \cdot 1,\ 4 \cdot 2,\ 4 \cdot 3])$ given in Example 1.3.1 has the corresponding array:

$$\begin{array}{c|cccccccccccc}
 & a_0 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 & c_0 & c_1 & c_2 & c_3 \\
\hline
a_0 & \phi & \phi & \phi & \phi & c_0 & \phi & \phi & \phi & b_0 & \phi & \phi & \phi \\
a_1 & \phi & \phi & \phi & \phi & \phi & c_1 & \phi & \phi & \phi & b_1 & \phi & \phi \\
a_2 & \phi & \phi & \phi & \phi & \phi & \phi & c_2 & \phi & \phi & \phi & b_2 & \phi \\
a_3 & \phi & \phi & \phi & \phi & \phi & \phi & \phi & c_3 & \phi & \phi & \phi & b_3 \\
b_0 & c_0 & \phi & \phi & \phi & \phi & \phi & \phi & \phi & a_0 & \phi & c_3 & c_2 \\
b_1 & \phi & c_1 & \phi & \phi & \phi & \phi & \phi & \phi & c_3 & a_1 & \phi & c_0 \\
b_2 & \phi & \phi & c_2 & \phi & \phi & \phi & \phi & \phi & c_1 & c_0 & a_2 & \phi \\
b_3 & \phi & \phi & \phi & c_3 & \phi & \phi & \phi & \phi & \phi & c_2 & c_1 & a_3 \\
c_0 & b_0 & \phi & \phi & \phi & a_0 & c_3 & c_1 & \phi & \phi & b_2 & b_1 & \phi \\
c_1 & \phi & b_1 & \phi & \phi & \phi & a_1 & c_0 & c_2 & b_2 & \phi & b_3 & \phi \\
c_2 & \phi & \phi & b_2 & \phi & c_3 & \phi & a_2 & c_1 & \phi & b_3 & \phi & b_0 \\
c_3 & \phi & \phi & \phi & b_3 & c_2 & c_0 & \phi & a_3 & b_1 & \phi & b_0 & \phi
\end{array}$$

Figure 1 Steiner Square from Example 1.3.1

1.5 Construction of a 3-PSTS $([\boldsymbol{n \cdot n, n \cdot n, n \cdot n}])$

Theorem 1.3 *There exists a 3-PSTS* $([n \cdot n, n \cdot n, n \cdot n])$ $\forall\ n \geq 1$ *with* $N_{111} = n^2$.

Proof: Let $A = \{a_x : x \in \mathbb{Z}_n\}$, $B = \{b_y : y \in \mathbb{Z}_n\}$, $C = \{c_z : z \in \mathbb{Z}_n\}$, $\mathcal{X} = A \cup B \cup C$ and $\mathcal{T} = \{\{a_x, b_y, c_z\} : x + y + z = 0, x, y, z \in \mathbb{Z}_n\}$.

Then, because for each $u, v \in \mathbb{Z}_n$, there is a unique $w \in \mathbb{Z}_n$ such that $u + v + w = 0$. It follows that

1. $(\mathcal{X}, \mathcal{T})$ is a PSTS.
2. If $T \in \mathcal{T}$, then $deg_{\mathcal{T}}(x) = n$ for all $x \in \mathcal{X}$.
3. $N_{111} = n^2$.
4. No triple is completely contained in any class.

Hence, $(\mathcal{X}, [A, B, C], \mathcal{T})$ is a 3-PSTS $([n \cdot n, n \cdot n, n \cdot n])$ for all $n \geq 1$ with $N_{111} = n^2$.
■

1.6 Construction of a 3-PSTS $([\boldsymbol{n \cdot x, n \cdot x, n \cdot x}]), \boldsymbol{0 \leq x \leq n}$

A *Group divisible design* $(k, \lambda) - \text{GDD}(h^u)$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a finite set of size $v = hu$, $\mathcal{G}$ is a partition of $\mathcal{V}$ into u *groups* each containing h elements, $\mathcal{B}$ is a collection of k element subsets of $\mathcal{V}$ called *blocks* that satisfy:

- If $B \in \mathcal{B}$, then $|B| = k$.
- If a pair of elements from $\mathcal{V}$ appear in the same group, then the pair cannot be in any block.
- Two points that are not in the same group, called a *transverse pair*, appear in exactly λ blocks.
- $|\mathcal{G}| > 1$.

A *resolvable GDD* (RGDD) has the additional condition that the blocks can be partitioned into parallel classes such that each element of $\mathcal{V}$ appears exactly once in each parallel class. A *uniform* RGDD is one in which $\lambda = 1$. We denote such an RGDD as a $k-$RGDD (h^u). For the purposes of this thesis, we will only talk about uniform RGDDs.

The necessary condition for the existence of a $k-$RGDD (h^u) was proved by L. Zhu in (1993).

Theorem 1.4 [3] *The necessary conditions for the existence of a* $k-$*RGDD* (h^u) *are:*

1. $u \geq k$,
2. $hu \equiv 0\ (mod\ k)$,
3. $h\ (u - 1) \equiv 0\ (mod\ k - 1)$.

Sufficient conditions for the existence of (k, λ) −RGDD (h^u)s have been discovered for $k = 2, 3, 4$ and $\lambda = 1$ except in a finite number of cases.

Theorem 1.5 [4] *A* $(3, \lambda) -$ RGDD (h^u) *exists if and only if* $u \geq 3$, $\lambda h\ (u-1)$ $is\ even$, $hu \equiv 0\ (mod\ 3)$, and $(\lambda, h, u) \notin \{\ (1, 2, 6), (1, 6, 3)\} \cup \{\ (2j+1, 2, 3), (4j+2,\ 1,\ 6) : j \geq 0\}$.

Now we prove the next theorem.

Theorem 1.6 If n is not 2 or 6 then 3-PSTS $([n \cdot x, n \cdot x,\ n \cdot x])$ exists if and only if $0 \leq x \leq n$.

Proof: For such n, there exists there exists a resolvable 3-GDD of type n^3 whose blocks can be partitioned into parallel classes $P_1, P_2, \cdots, P_n$. Take x of the parallel classes to form a 3-PSTS $([n \cdot x, n \cdot x, n \cdot x])$. ■

2. UNIFORM AND DENSE

It is obvious that in a Steiner square $([n \cdot \alpha, n \cdot \beta, n \cdot \gamma])$, the entry $\mathrm{M}[x, x] = \emptyset$ for every $x \in A \cup B \cup C$. The number of non-empty cells in a $3n \times 3n$ Steiner square $([n \cdot \alpha, n \cdot \beta, n \cdot \gamma])$ is $2(\alpha n + \beta n + \gamma n)$ while the total possible number of filled cells is $(3n)^2 - 3n$. Thus, a necessary condition for the existence of a 3-PSTS $([n \cdot \alpha, n \cdot \beta,\ n \cdot \gamma])$ is $2(\alpha n + \beta n + \gamma n) \leq (3n)^2 - 3n$ *or* $2(\alpha + \beta + \gamma) \leq 3\,(3n - 1)$.

In a Steiner square if every cell is filled except the entries $\mathrm{M}[x, x]$ where $x \in A \cup B \cup C$, then

$$2(\alpha + \beta + \gamma) = 3\,(3n - 1) \tag{3}$$

Thus, *n* must be odd.

Suppose $(\mathcal{X},\ [A, B, C],\ \mathcal{T})$ is a 3-PSTS $([n \cdot \alpha, n \cdot \beta,\ n \cdot \gamma])$. Counting triples that contain a point x that has degree α, we obtain the following equation:

$$N_{120} + N_{102} + N_{111} + 2N_{210} + 2N_{201} = n\alpha \tag{4}$$

Similarly, counting triples that contain a point y that has degree β and triples that contain a point z that has degree γ, we get two more equations:

$$2N_{021} + 2N_{120} + N_{210} + N_{012} + N_{111} = n\beta \tag{5}$$

$$N_{111} + 2N_{102} + N_{201} + 2N_{012} + N_{021} = n\gamma \tag{6}$$

Furthermore, counting triples that contain pairs (x, y) such that $\deg(x) = \alpha$ and $\deg(y) = \beta$, we get:

$$N_{111} + 2N_{120} + 2N_{210} \leq n^2 \tag{7}$$

Similarly, counting triples that contain pairs (x, z) such that $\deg(x) = \alpha$ and $\deg(z) = \gamma$; and triples that contain pairs (y, z) such that $\deg(y) = \beta$ and $\deg(z) = \gamma$ gives:

$$N_{111} + 2N_{201} + 2N_{102} \leq n^2 \tag{8}$$

$$N_{111} + 2N_{012} + 2N_{021} \leq n^2 \tag{9}$$

Finally, counting the total number of triples gives the last equation:

$$N_{120} + N_{111} + N_{102} + N_{210} + N_{021} + N_{012} + N_{201} = \frac{n\alpha + n\beta + n\gamma}{3} \tag{10}$$

We define a *dense* 3-PSTS as the 3-PSTS where every transverse pair is covered. Thus, a dense 3-PSTS achieves the equality in Equations (7), (8) and (9). Now solving these 7 equations, in terms of N_{021} we have:

$$
\begin{aligned}
N_{120} &= -N_{021} + \beta n - n^2 \\
N_{111} &= \frac{1}{3}\,(-2\alpha n - 2\beta n - 2\gamma n + 9n^2) \\
N_{102} &= +N_{021} + \frac{1}{3}(-\alpha n - \beta n + 2\gamma n) \\
N_{210} &= +N_{021} + \frac{1}{3}(\,\alpha n - 2\beta n + \gamma n) \\
N_{021} &= N_{021} \\
N_{012} &= -N_{021} + \frac{1}{3}(\alpha n + \beta n + \gamma n - 3n^2) \\
N_{201} &= -N_{021} + \frac{1}{3}\,(2\alpha n + 2\beta n - \gamma n - 3n^2)
\end{aligned}
\tag{11}
$$

If we fix the value of N_{021} to be zero, then we have:

$$
\begin{aligned}
N_{120} &= \beta n - n^2 \\
N_{111} &= \frac{1}{3}\,(-2\alpha n - 2\beta n - 2\gamma n + 9n^2) \\
N_{102} &= \frac{1}{3}(-\alpha n - \beta n + 2\gamma n) \\
N_{210} &= \frac{1}{3}(\,\alpha n - 2\beta n + \gamma n) \\
N_{021} &= 0 \\
N_{012} &= \frac{1}{3}(\alpha n + \beta n + \gamma n - 3n^2) \\
N_{201} &= \frac{1}{3}\,(2\alpha n + 2\beta n - \gamma n - 3n^2)
\end{aligned}
\tag{12}
$$

Many of the constructions used in this thesis rely on standard techniques that are known as *difference techniques* [5]. A $(v, k, \lambda) -$ BIBD is a design $(\mathcal{X}, \mathcal{A})$ such that the following properties are satisfied.

1. $|\mathcal{X}| = v$
2. Each block has k points.
3. Every pair of distinct points is contained in exactly λ blocks.

In a $(v, k, \lambda) -$ BIBD, a *difference set or base block* [6] is a subset D of size k of a group $(Z_v, +)$ such that every element of $Z_v \setminus \{0\}$ can be expressed as a difference of two elements of D in exactly λ ways. For any $g \in Z_v$, define $D + g = \{x + g : x \in D\}$. The set $D + g$ is called a *translate* of D, and we refer to the collection of all v translates of D as the development of D, denoted by *Dev(D)*. In the following example, we use the technique of developing base blocks to construct designs.

2.1 Example: A (13, 4, 1) – BIBD

The differences in $\mathbb{Z}_{13}$ are $\{\pm1, \pm2, \pm3, \pm4, \pm5, \pm6\}$. Let D = {0, 1, 3, 9} be a base block. Notice the differences in this block are:

1 – 0 = 1	0 – 1 = 12
3 – 1 = 2	1 – 3 = 11
3 – 0 = 3	0 – 3 = 10
0 – 9 = 4	9 – 0 = 9
1 – 9 = 5	9 – 1 = 8
9 – 3 = 6	3 – 9 = 7

The development of D consists of the following blocks: { 1, 2, 4, 10}, {2, 3, 5, 11}, {3, 4, 6, 12}, {4, 5, 7, 0}, {5, 6, 8, 1}, {6, 7, 9, 2}, {7, 8, 10, 3}, {8, 9, 11, 4}, {9, 10, 12, 5}, {10, 11, 0, 6}, {11, 12, 1, 7}, {12, 0, 2, 8}. Any difference, d, in the base block D, is defined by some pair (x, y) where $x - y = d$. In *Dev(D)*, $\{x + g,\ y + g\}$ also has difference d for each $g \in \mathbb{Z}_{13}$. Thus, every edge of difference d appears exactly once in *Dev(D)*. This gives us a (13, 4, 1) – BIBD. ■

In our constructions of 3-PSTS, we will refer to two specific types of differences, cross differences and side differences. We define a *cross difference* as the difference between two points belonging to two different classes while a *side difference* is the difference covered between two points belonging to same class. Specifically, if $(\mathcal{X}, [A, B, C], \mathcal{T})$ is a 3-PSTS ($[n \cdot \alpha, n \cdot \beta, n \cdot \gamma]$), then we define the cross difference between any pair (x, y) such that $x \in A, y \in B$; $x \in B, y \in C$ or $x \in C, y \in A$ as $y - x \in \mathbb{Z}_n$. The side difference between any pair (x, y) such that both x and y are in the same class defined as $y - x \in \mathbb{Z}_n$. Note that when n is even, the side difference of $\frac{n}{2}$ has the property that $\frac{n}{2} \equiv -\frac{n}{2}\ mod\ (n)$. Thus, if we create a base triple covering the side difference $\frac{n}{2}$, the development will contain repeated pairs which are covered in the first $\frac{n}{2}$ translates and then repeated in the last $\frac{n}{2}$ translates.

2.2 Steiner triple systems of order $(6x + 3)$

Suppose $(\mathcal{X},\ [A, B, C],\ \mathcal{T})$ is a dense 3-PSTS ($[n \cdot \alpha, n \cdot \alpha,\ n \cdot \alpha]$). If we assume every cell of the Steiner square to be filled except the diagonal, then according to equation (3) $\alpha = \frac{3n-1}{2}$ and n is odd. Therefore, $n \equiv 1,3,5 \pmod 6$ which implies that $3n \equiv 3 \pmod 6$. Then, because $\alpha = \frac{3n-1}{2}$ is exactly equal to the replication number of a $STS(3n)$, it follows that it may be possible for us to construct an $STS(3n)$ in such a way that it is also a dense 3-PSTS ($[n \cdot \alpha, n \cdot \alpha,\ n \cdot \alpha]$). We call this a 3-STS

$([n \cdot \alpha, n \cdot \alpha, n \cdot \alpha])$. Let $3n = 6x + 3$ so that $n = 2x + 1$. We can determine the number of triples of each type that a 3-PSTS $([n \cdot \alpha, n \cdot \alpha, n \cdot \alpha])$ must have by using the seven equations.

$$
\begin{aligned}
N_{120} &= n\,(\alpha - n) = x\,(2x+1) \\
N_{111} &= n(3n - 2\alpha) = n = 2x + 1 \\
N_{102} &= 0 \\
N_{210} &= 0 \\
N_{021} &= 0 \\
N_{012} &= n\,(\alpha - n) = x\,(2x+1) \\
N_{201} &= n\,(\alpha - n) = x(2x+1)
\end{aligned}
\tag{13}
$$

We construct a 3-PSTS $([n \cdot \alpha, n \cdot \alpha, n \cdot \alpha])$ as follows.
Partition the $3n = 6x + 3$ points into three disjoint classes A, B and C, each of size $2x + 1$.

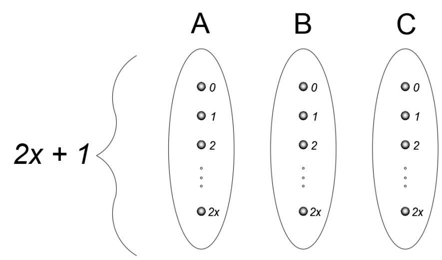

Figure 2 Partition of points into three classes

Because this design is a $STS(3n)$, we must cover every pair exactly once. Thus, between each pair of classes, we must cover cross differences $0, 1, 2, \cdots, 2x$ while within each class we want to cover the side differences $\pm 1, \pm 2, \pm 3, \cdots, \pm x$. We begin by constructing Type (120) triples. In order to construct Type (120) triples, we pick cross difference pairs which sum to $2x + 1$ as shown.

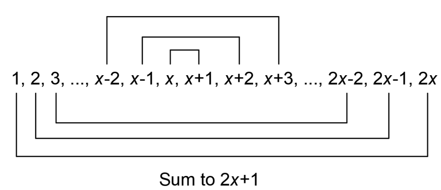

Figure 3 Cross difference pairs

Let (b_1, b_2) be a cross difference pair. Then, form the base blocks $\{0, b_1, b_2\}$ where $0 \in A$ and $b_1, b_2 \in B$. Because the cross difference pairs each sum to $2x + 1$, we have that $\{\, b_2 - b_1 \pmod{2x+1} : (b_1, b_2) \text{ is a cross difference pair}\} = \{1, 2, 3, \cdots, x\}$.

Furthermore, because the set of cross difference pairs partition the set $\{1, 2, 3, \cdots, 2x\}$, the base blocks $\{0, b_1, b_2\}$ cover all cross differences between A and B. We give the cross difference pairs and the corresponding base blocks in Table 1.

Table 1 Base blocks corresponding to cross difference pairs

Base block	Side difference $(\pm)$	Cross differences
$0, x, x+1$	1	$x, x+1$
$0, x-1, x+2$	3	$x-1, x+2$
$0, x-2, x+3$	5	$x-2, x+3$
:	:	:
:	:	:
$0, 2, 2x-1$	4	$2, 2x-1$
$0, 1, 2x$	2	$\mathbf{1, 2x}$

Now take the development of each base block to get a total of $N_{120} = x\ (2x+1)$ triples. Similarly, Type (012) and Type (201) triples can be constructed in the same way covering all side differences and cross differences between classes B and C and classes C and A respectively. The total number of triples we have constructed so far is $x\ (\ 2x+1) + x\ (2x + 1) + x\ (2x+1) = 3x\ (2x+1)$. We can construct $2x+1$ Type (111) triples by developing the base block $\{0, 0, 0\}$ which covers the cross difference 0 between each pair of classes. Thus the total number of triples constructed is $3x\ (2x+1) + (2x+1) = (3x + 1)(2x+1)$, which is exactly the total number of triples in an $STS(6x+3)$. Furthermore, each point in class A occurs in x Type (120) triples, $2x$ Type (201) triples and 1 Type (111) triple. Hence, degree of each point in class A = $x + 2x + 1 = 3x + 1$. Similarly, each point in class B occurs in x Type (012) triples, $2x$ Type (120) triples and 1 Type (111) triple. So, the degree of each point in class B is $x + 2x + 1 = 3x + 1$. Each point in class C occurs in $2x$ Type (012) triples, x Type (201) triples and 1 Type (111) triple. Hence, the degree of each point in class C is $2x + x + 1 = 3x + 1$. Therefore, the degree of each point in this system is $3x + 1$ which is the replication number of each point in an $STS(6x+3)$.

Thus, we have the following result.

Theorem 2.1 *For every $x \geq 1$, there exists a uniform 3–STS* $([2x+1 \cdot 3x+1, 2x+1 \cdot 3x+1, 2x+1 \cdot 3x+1])$.

■

2.2.1 Example: Construction of a 3-STS $([5 \cdot 7, 5 \cdot 7, 5 \cdot 7])$

When $n = 5$ and $\alpha = 7$, the equations in (13) give $N_{120} = N_{012} = N_{201} = 10$ and $N_{111} = 5$. Since we are working over $\mathbb{Z}_5$, we must cover cross differences {0, 1, 2, 3, 4} and side differences {±1, ±2}. While constructing Type (120) triples, we pick cross difference pairs (2, 3) and (1, 4) to cover all the differences between class A and B while the corresponding side differences covered in class B are 1 and 2 respectively. This has been illustrated in the Figure 4.

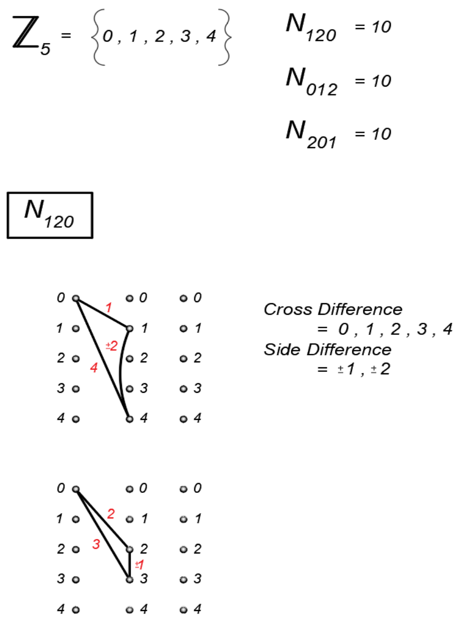

Figure 4 Construction of Type (120) triples in Example 2.2.1

Now developing these two base triples (mod 5), we get 10 Type (120) triples. Then, we can construct Type (012) and Type (201) triples in a similar way. See Figure 5.

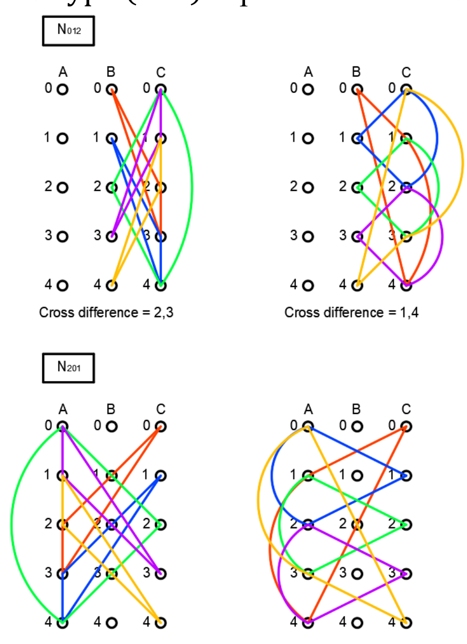

Figure 5 Construction of Type (012) and (201) triples in Example 2.2.1

Type (111) triples can be constructed trivially covering cross-difference 0 between every pair of classes by developing the base block $\{0, 0, 0\}$. This is illustrated in Figure 6.

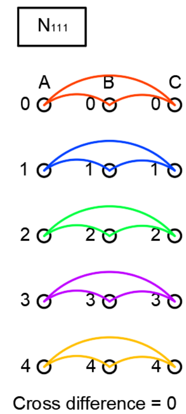

Figure 6 Construction of Type (111) triples in Example 2.2.1

2.3 The Bose Construction

A *Latin square* of order n is an $n \times n$ array with each cell containing exactly one of the symbols $\{1, 2, \cdots, n\}$ in such a way that each row and each column covers each of the symbols in $\{1, 2, \cdots, n\}$ exactly once. A Latin square is said to be *Idempotent* if cell (i, i) contains symbol i for $1 \leq i \leq n$. A Latin square is said to be *Commutative or symmetric* if the cell (i, j) and (j, i) contain same symbol for $1 \leq i, j \leq n$. Suppose o is a binary operation defined on a set Q of cardinality 'n'. We define (Q, o) to be a *Quasigroup* if and only if its operation table is a Latin square of order 'n'.

Theorem 2.2 [5] There exists a symmetric idempotent Quasigroup of order n if and only if n is odd.

In (1939), Bose found a construction for $STS(6x + 3)$ when $x \geq 1$. The Bose Construction [7] uses commutative idempotent Latin squares to construct $STS(6x + 3)$. Let $v = 2x + 1$ and let (Q, o) be an idempotent commutative quasigroup of order $2x + 1$. Let $Q = \{0, 1, 2, \cdots, 2x\}$. We denote $W = Q \times \{0, 1, 2\}$ and $\mathcal{T}$ to be set of triples of two types:

Type 1: For all $0 \leq i, j \leq 2x$, $\{(i, 0), (j, 0), (i \; o \; j, 1)\}$, $\{(i, 1), (j, 1), (i \; o \; j, 2)\}$, $\{(i, 2), (j, 2), (i \; o \; j, 0)\}$

Type 2: For all $0 \leq i \leq 2x$, $\{(i, 0), (i, 1), (i, 2)\}$

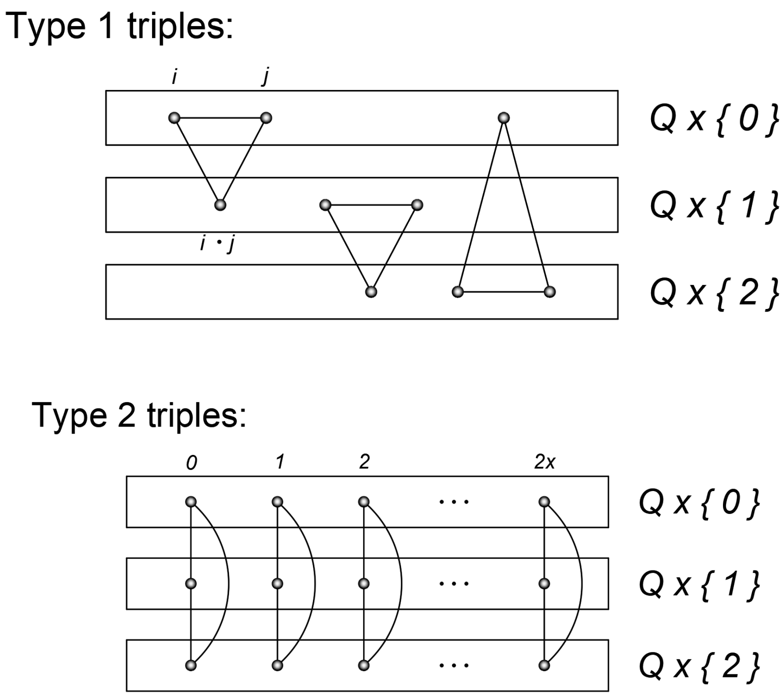

Figure 7 Bose Construction

From the figure, it is clear that Type 1 triples are of Type (210), $(021)$and (102) while Type 2 triples are of type (111). Clearly, we have $2x+1$ Type (111) triples. Again, $N_{210} = N_{021} = N_{102} = \frac{(2x+1)(2x)}{2} = x(2x+1)$.This suggests that the above construction generates a uniform 3-STS. While the Bose Construction uses Latin squares for the construction (Figure 7), we were able to construct it using the method of differences.

3. UNIFORM ONLY

This section is for investigating the uniform 3- PSTS $([n \cdot \alpha\ ,\ n \cdot \beta,\ n \cdot \gamma])$ that are not necessarily dense. Thus, we construct 3- PSTS with $2(\alpha + \beta + \gamma) \leq 3\ (3n - 1)$.

We begin with the case $\alpha = \beta = \gamma$. We already saw that a dense $([n \cdot \alpha,\ n \cdot \alpha,\ n \cdot \alpha])$ is possible only when n is odd and $\alpha = \frac{3n-1}{2}$. Then for any uniform 3- PSTS $([n \cdot \alpha\ ,\ n \cdot \alpha,\ n \cdot \alpha])$ (dense or not dense), we have $\alpha \leq \frac{3n-1}{2}$. When $n = 2k,$ we have $\alpha \leq \frac{3n-1}{2} = \frac{3(2k)-1}{2} = \lfloor 3k - \frac{1}{2} \rfloor = 3k - 1$. We would like to answer the following question: For what values of $n = 2k,$ can we achieve the maximum possible degree?

3.1 Example: Construction of a 3- PSTS $([\mathbf{8 \cdot 10, 8 \cdot 10, 8 \cdot 10}])$

We follow the same construction method used to prove Theorem 2.1. Note that we cannot cover the side difference ± 4 in our construction because pairs would be repeated in the development. So, we will pick cross difference pairs in such a way that we will cover all side differences except ± 4. We begin with the construction of Type (120) triples. Then Type (012) and Type (201) triples can be constructed by the same process. The cross difference pairs (1, 7), (2, 5) and (3, 4) cover the side differences $\pm 2, \pm 3$ and ± 1 respectively. Note that we did not cover the cross difference '6'.

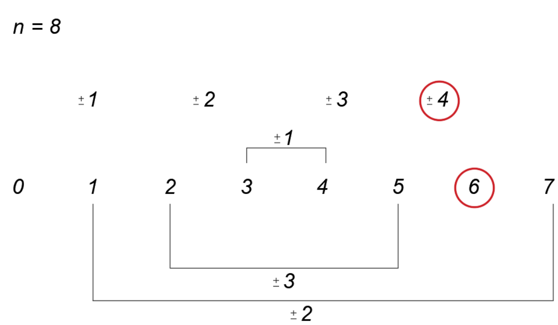

Figure 8 Cross difference pairs for Example 3.1

We construct Type (120) triples corresponding to each cross difference pair by developing the corresponding base triples (mod 8) as shown in Figure 9. The base triples are marked in 'red' and the other triples obtained by the development are marked in 'black'.

N_{120}

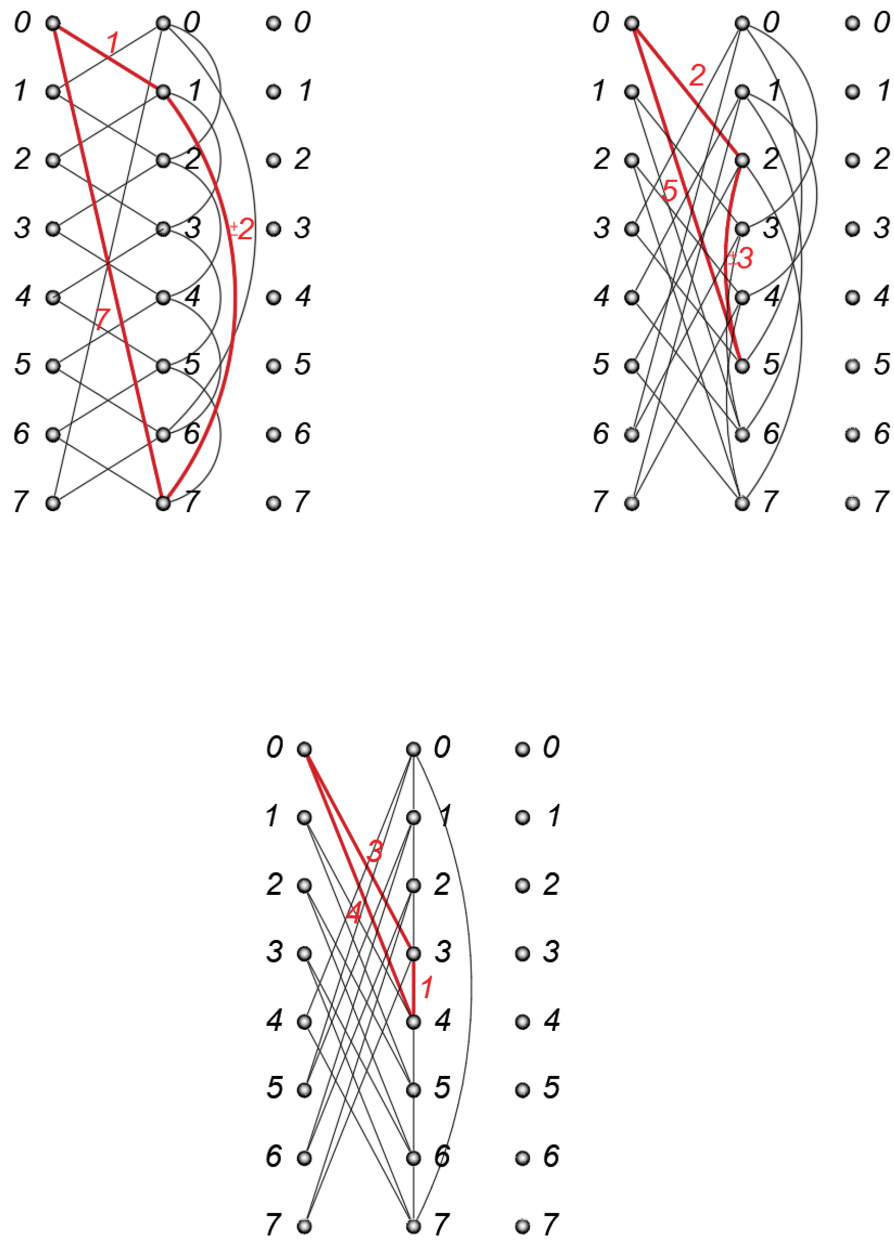

Figure 9 Construction of Type (120) triples in Example 3.1

Type $(012)$and (201) triples can also be constructed in the same manner. Furthermore, Type (111) triples can be constructed by developing the base block $\{0,0,0\}$. Since there are 3 complete sets of (120) triples, (012) triples and (201) triples, and 1 complete set of (111) triples, the degree of each class is $3\ (3) + 1 = 10$. Since $n = 8$ and $k = 4$, with this construction we could achieve a 3- PSTS $([n \cdot \alpha\ ,\ n \cdot \alpha,\ n \cdot \alpha])$ with $\alpha = 3k - 2 = 3(4) - 2 = 10$. Our goal is to find constructions in which α is as close to $3k - 1$ as possible.

We generalize this example in the following construction.

3.2 Case 1: $n = 2k, k\ even$

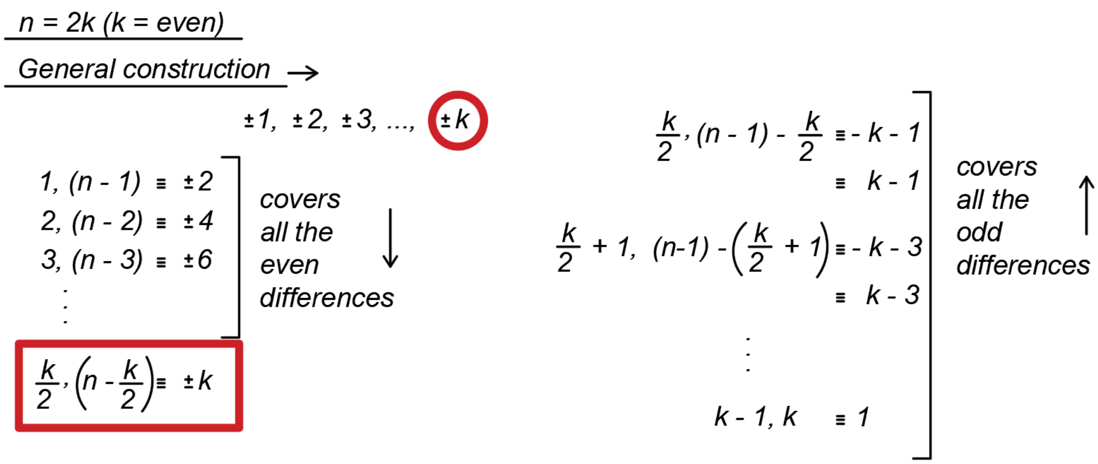

Figure 10 3-PSTS$([2k \cdot (3k - 2), 2k \cdot (3k - 2), 2k \cdot (3k - 2)])$: k even

The main part of this construction is to describe how to pick the cross difference pairs from the set $\{1, 2, 3,\cdots, 2k\}$. We choose pairs $(1, n - 1), (2, n - 2), (3, n - 3) \cdots$ such that the elements in each pair sum to $n = 2k$ and we continue like this until we reach the pair $(k/2, n\text{-}(k/2))$. As mentioned earlier, we cannot select this pair because it covers the side difference $\pm k$. Hence, we pick the pair $(k/2, (n - 1)\text{-} k/2)$ and continue so that the elements in each pair sum to $n - 1$. From the Figure 10, it can be seen that the cross difference pairs summing to n cover all the even side differences of the set $\{1, 2, 3,\cdots, k\}$ except $\pm k$ while those pairs summing to $n - 1$ cover all the odd side differences of the same set. Since each pair of cross difference has a corresponding side difference, we can construct Type (120), (012) and (201) triples for each pair. The cross difference pairs and the corresponding base blocks are shown in Table 2 and Table 3.

Table 2 Base blocks corresponding to even side differences in Case-1

Base block	Side difference $(\pm)$	Cross differences
$0,1,n-1$	2	$1,n-1$
$0,2,n-2$	4	$2,n-2$
$0,3,n-3$	6	$3,n-3$
:	:	:
:	:	:
$0,\ \frac{k}{2}-1,\ n-\left(\frac{k}{2}-1\right)$	$n-2\left(\frac{k}{2}-1\right)=k-2$	$\frac{k}{2}-1,n-\left(\frac{k}{2}-1\right)$

Table 3 Base blocks corresponding to odd side differences in Case-1

Base block	Side difference $(\pm)$	Cross differences
$0,\frac{k}{2},(n-1)-\frac{k}{2}$	$k-1$	$\frac{k}{2},(n-1)-\frac{k}{2}$
$0,\frac{k}{2}+1,(n-1)-\left(\frac{k}{2}+1\right)$	$k-3$	$\left(\frac{k}{2}+1\right),(n-1)-\left(\frac{k}{2}+1\right)$
:	:	:
:	:	:
$0,\ k-1,\ k$	1	$k-1,k$

From Figure 10, it is clear that there are $\frac{k}{2}-1$ pairs covering the even differences and $\frac{k}{2}$ pairs covering all the odd differences. Hence, the total number of pairs is $\frac{k}{2}-1+\frac{k}{2}=k-1$. Then, we can construct $k-1$ sets of Type (120), (012) and (201) triples and 1 complete set of Type (111) triples by developing the base block $\{0,0,0\}$. Therefore, the degree of each vertex in each class is $3\ (k-1)+1=3k-2$. We do not cover the side difference of $\pm k$ and the cross difference of $n-\frac{k}{2}=\frac{3n}{4}$ in this case.

Similarly, we can construct the 3-PSTS $([2k\ \cdot(3k-2),2k\ \cdot(3k-2),2k\ \cdot(3k-2)])$ for odd k.

3.3 Case 2: $n = 2k, k$ *odd*

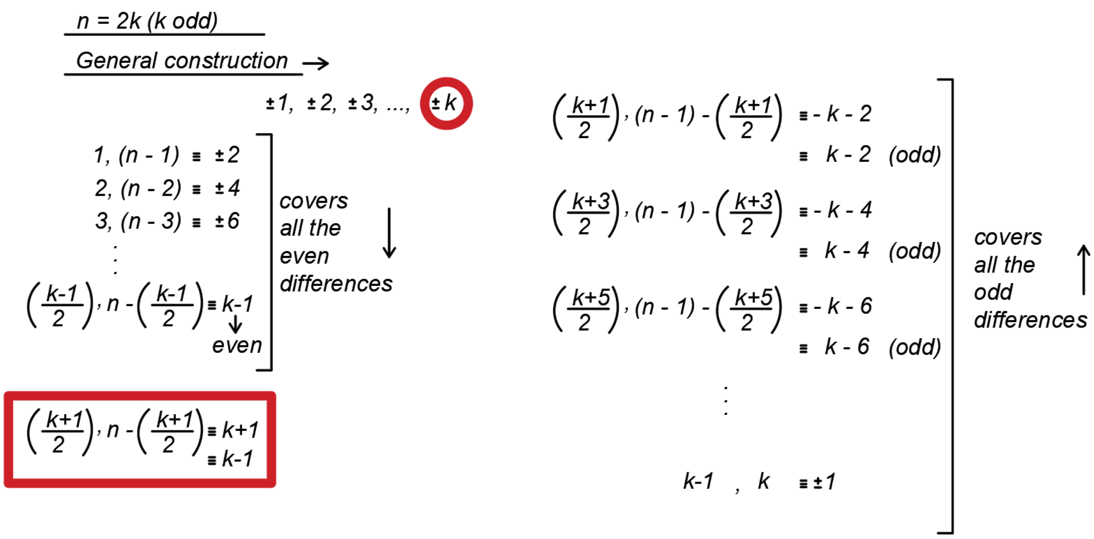

Figure 11 3-PSTS($[2k \cdot (3k-2), 2k \cdot (3k-2), 2k \cdot (3k-2)]$): k odd

Similar to the previous construction, we first start picking cross difference pairs $(1, n-1), (2, n-2), (3, n-3) \cdots$ that sum to $n = 2k$. We keep going until we reach the pairs $(\frac{k-1}{2}, n - \frac{k-1}{2})$ and $(\frac{k+1}{2}, n - \frac{k+1}{2})$. However, both of these pairs cover same side difference $k-1$ which cannot be allowed. Hence, considering the former pair and discarding the latter, we pick the pair $\left(\frac{k+1}{2}, (n-1) - \frac{k+1}{2}\right)$ and continue such that the elements in each pair sum to $n-1$. In this way, the cross difference pairs summing to n cover all the even side differences except $\pm k$ while the pairs summing to $n-1$ cover all the odd differences of the set $\{1, 2, 3, \cdots, k\}$. The cross difference pairs and the corresponding base blocks are shown in the tables below.

Table 4 Base blocks corresponding to even side differences in Case-2

Base block	Side difference $(\pm)$	Cross differences
$0, 1, n-1$	2	$1, n-1$
$0, 2, n-2$	4	$2, n-2$
$0, 3, n-3$	6	$3, n-3$
:	:	:
:	:	:
$0, \frac{k-1}{2}, n - \left(\frac{k-1}{2}\right)$	$n - 2\left(\frac{k-1}{2}\right) = k-1$	$\frac{k-1}{2}, n - \left(\frac{k-1}{2}\right)$

Table 5 Base blocks corresponding to odd side differences in Case-2

Base block	Side difference $(\pm)$	Cross differences
$0, \frac{k+1}{2}, (n-1) - \frac{k+1}{2}$	$k-2$	$\frac{k+1}{2}, (n-1) - \frac{k+1}{2}$
$0, \frac{k+3}{2}, (n-1) - \frac{k+3}{2}$	$k-4$	$\frac{k+3}{2}, (n-1) - \frac{k+3}{2}$
$0, \frac{k+5}{2}, (n-1) - \frac{k+5}{2}$	$k-6$	$\frac{k+5}{2}, (n-1) - \frac{k+5}{2}$
:	:	:
:	:	:
$0,\ k-1,\ k$	1	$k-1, k$

Then, it is easy to construct Type $(120), (012)$ and (201) triples for each cross difference pair. From Figure 11, we can see that there are $\frac{k-1}{2}$ pairs covering the even differences and $\frac{k-1}{2}$ pairs covering all the odd differences. Hence, the total number of pairs is $\frac{k-1}{2} + \frac{k-1}{2} = k-1$. Now we have $k-1$ sets of Types $(120), (012)$and (201) triples and 1 complete set of (111) triples (covering the cross difference of 0 across three classes). Therefore, the degree of each class is $3\ (k-1) + 1 = 3k-2$. Note: We do not cover the side difference of $\pm k$ and the cross difference of $n - \frac{k+1}{2} = \frac{3n-2}{4}$ in this case.

Combining Case 1 and Case 2, we have the following theorem.

Theorem 3.1 *There exists a 3-PSTS* $([2k\ \cdot (3k-2), 2k\ \cdot (3k-2), 2k\ \cdot (3k-2)])$ *for all positive integers* k. ■

From Theorem 3.1, we learned that for every k, we can construct a 3-PSTS $([2k\ \cdot (3k-2), 2k\ \cdot (3k-2), 2k\ \cdot (3k-2)])$. We would like to determine for which values of k we can construct a 3-PSTS $([2k\ \cdot (3k-1), 2k\ \cdot (3k-1), 2k\ \cdot (3k-1)])$, since $3k-1$ is the maximum bound on α.

3.4 Example: Construction of a dense 3- PSTS $([6 \cdot 8, 6 \cdot 8, 6 \cdot 8])$

Based on the construction given for the proof of Theorem 3.1, we can get a uniform 3-PSTS $([6 \cdot 7, 6 \cdot 7, 6 \cdot 7])$ by using cross difference pairs $(1, 5)$ and $(2, 3)$ which cover side differences ± 2 and $\pm\ 1$ respectively and by developing the base block {0, 0, 0} covering the cross difference 0 among the three classes. Note that the cross difference of 4 has not been covered. See Figure 12.

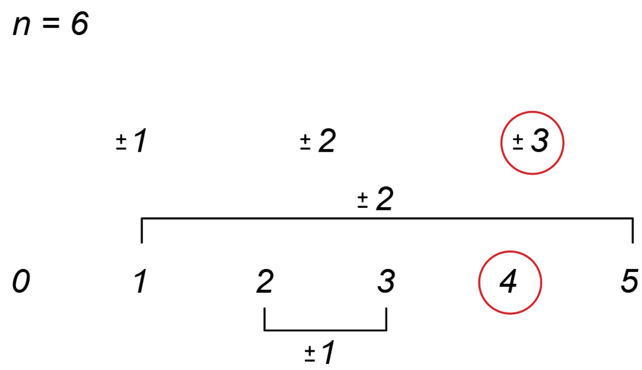

Figure 12 Cross difference and side differences in Example 3.4

However, we can obtain a set of Type (111) triples covering the cross difference 4 among all the three classes. The corresponding base block is shown in Figure 13.

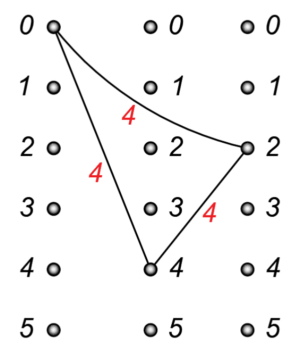

Figure 13 A base triple of Type (111) covering cross difference 4 in Example 3.4

Now developing this base triple modulo 6, we get a complete set of Type (111) triples which is shown in Figure 14.

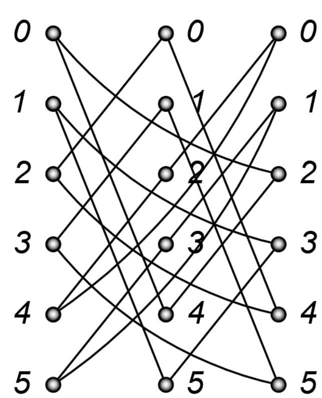

Figure 14 Complete set of Type (111) triples developed from Figure 13

These additional sets of Type (111) triples add one more degree to the vertices in each class. Hence, the degree of each vertex is 2 (3) + 1 + 1 = 8. Since all the transverse pairs are covered, this is a dense 3-PSTS $([6 \cdot 8, 6 \cdot 8, 6 \cdot 8])$.

We see that $n = 6$ is a special case where we can achieve the maximum possible degree $\alpha = 3k - 1 = 8$. This example inspires us to ask if we can determine when it is possible to find a uniform 3-PSTS with α as large as possible. In the next section, we will discuss how Skolem sequences can be used to find such uniform dense 3- PSTSs.

3.5 Using Skolem Sequence

A Skolem sequence of order n is a sequence $S = (s_1, s_2, \cdots, s_{2n})$ of $2n$ integers satisfying the following conditions:

- For all $k \in \{1, 2, \cdots, n\}$ there are two elements s_i and $s_j \in S$ such that $s_i = s_j = k$.
- If $s_i = s_j = k$ and $i < j$, then $j - i = k$.

3.5.1 Example: A Skolem sequence of order 5

$$S = (1,\ 1,\ 3,\ 4,\ 5,\ 3,\ 2,\ 4,\ 2,\ 5)$$

$$\underbrace{(1\,,\,2)}_{1} \quad \underbrace{(7\,,\,9)}_{2} \quad \underbrace{(3\,,\,6)}_{3} \quad \underbrace{(4\,,\,8)}_{4} \quad \underbrace{(5\,,\,10)}_{5}$$

An extended Skolem sequence of order n is a sequence ES = $(s_1, s_2, \cdots, s_{2n},\ s_{2n+1})$ of $2n + 1$ integers satisfying conditions 1 and 2 above and there is exactly one $s_i \in ES$ such that $s_i = 0$.

The element $s_i = 0$ is the $hook$ or $zero$ of the sequence.

3.5.2 Example: An Extended Skolem sequence of order 6

$$ES = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$$

$$\underbrace{(1\,,\,2)}_{1} \quad \underbrace{(3\,,\,5)}_{2} \quad \underbrace{(8\,,\,11)}_{3} \quad \underbrace{(6\,,\,10)}_{4} \quad \underbrace{(4\,,\,9)}_{5} \quad \underbrace{(7\,,\,13)}_{6}$$

The 12^{th} position is not involved in any pair, so 12 is the $hook$ of the sequence.

In (1981) Abraham and Kotzig proved in [8] that an extended Skolem sequence exists for all n.

Theorem 3.2 [8] *An extended Skolem sequence of order* n *exists for all* n.

In (1995), Baker provided the following result in [9].

Theorem 3.3 [9] *An extended Skolem sequence of order* n *exists for all positions* i *of the zero if and only if* i *is odd and* $n \equiv 0, 1 \ (mod\ 4)$ *or* i *is even and* $n \equiv 2, 3 \ (mod\ 4)$.

We will use extended Skolem sequences to prove the next result. We begin by describing two constructions for finding sets of triples of particular types. These constructions are generalizations of the constructions given in this work.

Construction 1 - Let A, B and C be three classes of size n with n even ($n = 2k$). Let X be a set of $(k-1)$ pairs (called cross pairs) on the set $Y \subseteq \{1, 2, \cdots, 2k-1\}$ such that:

- Every element in Y appears at most once among the $(k-1)$ pairs in X.
- For each pair $(x_1, x_2) \in$ X, the difference (mod n) of the pair $s = x_2 - x_1 \in \{\pm 1, \pm 2, \cdots, \pm(k-1)\}$.
- For any two pairs $(x_1, x_2), (x_1', x_2') \in$ X, we have $s \neq s'$.
- $\bigcup_{i=1}^{k-1} s_i = \{\pm 1, \pm 2, \cdots, \pm(k-1)\}$.

For each pair $(x_1, x_2) \in$ X, form the base blocks $(0, x_1, x_2)$ where $0 \in A, x_1 \in B, x_2 \in B$. This block uses the cross differences x_1 and x_2 between classes A and B. It also uses the side difference $\pm(x_2 - x_1)$ (mod n) on class B. Let D = $\{(0, x_1, x_2) | \ 0 \in A, \ x_1, x_2 \in B, (x_1, x_2) \in X\}$. Then *Dev(D)* produces a set of N_{120} triples of Type (120) which cover every cross difference in Y exactly once. They also cover every side difference in $\{\pm 1, \pm 2, \cdots, \pm(k-1)\}$ on B exactly once. Replicating this process on classes B and C, and on classes C and A produces a total of $3n(k-1)$ triples, with $N_{120} = N_{012} = N_{201} = n(k-1)$. ■

In Construction 1, every pair from distinct classes appears in at most one triple because $Y \subseteq \{1, 2, \cdots, 2k-1\}$. Every pair within any class also appears in at most one triple because the differences of the pairs in X are distinct. Furthermore, each vertex in Class A has degree $2(k-1) + (k-1) = 3(k-1)$ because each point in class A occurs in $(k-1)$Type (120) triples and $2(k-1)$Type (201) triples. Hence, the degree of each point in class A is $2(k-1) + (k-1) = 3(k-1)$. Similarly, each point in class B occurs in $k-1$ Type (012) triples and $2(k-1)$ Type triples. Each point in class C occurs in $2(k-1)$ Type (012) triples and $(k-1)$ Type (201) triples. Thus, the degree of each point in class B or class C is also $3(k-1)$. Thus, Construction 1 produces a 3-PSTS $([n \cdot 3(k-1), n \cdot 3(k-1), n \cdot 3(k-1)])$. ■

Construction 2 - Let $i \in \{0, 1, \cdots, n-1\} = \mathbb{Z}_n$ be an element of order 3 or 0. Then take $(0, \ i, \ 2i)$ to be a base block where $0 \in A, i \in B, 2i \in C$. Then this block covers the cross difference i between classes A and B, B and C, C and A. Then, $Dev(0, \ i, \ 2i)$ produces a set of n blocks of type (111) and contributes 1 to the degree of each vertex. ■

Theorem 3.4 *Let $n = 2k$ and $k \equiv 0,3$ $(mod\ 12)$. Then there exists a uniform dense 3-PSTS* $([2k \cdot (3k-1), 2k \cdot (3k-1), 2k \cdot (3k-1)])$.

Proof: Let $n = 2k$ and $k \equiv 0,3$ (mod 12). Then $k - 1 \equiv 2$ *or* 11(mod 12). So, $k - 1 \equiv 2$ *or* 3 (mod 4). By Theorem 3.3, there exists an extended Skolem sequence of order $k - 1$ for all positions i of the zero if i is even. Thus, there is a sequence on the set $\{1, 2, \cdots, 2k-1\} \backslash \{i\}$ consisting of $(k-1)$pairs that cover the differences $\chi = \{\pm 1, \pm 2, \cdots, \pm(k-1)\}$. Since $k \equiv 0,3$ (mod 12), it follows that $\frac{2k}{3}$ is an even integer, so we may choose $i = \frac{2k}{3}$. Now apply Construction 1 with $Y = \{1, 2, \cdots, 2k-1\} \backslash \{\frac{2k}{3}\}$ to obtain 3 sets of $n(k-1)$ triples such that the degree of every vertex is $3(k-1)$.

Because $i = \frac{2k}{3}$ has order 3 in $\mathbb{Z}_{2k}$ and $i = 0$ has order 0, we may apply Construction 2 to obtain two more sets of n triples of Type (111), each of which contribute 1 to the degree of each vertex. Therefore, each vertex now has degree $3\ (k-1) + 1 + 1 = 3k - 3 + 2 = 3k - 1$. ■

4. SUMMARY AND FUTURE WORK

Our ultimate objective was to find necessary and sufficient conditions for the existence of uniform 3-class PSTS. Motivated by the methods developed by Keranen, Kreher and Ozkan in [2], we were able to provide some new constructions of 3-PSTS ($[n \cdot \alpha, n \cdot \alpha, n \cdot \alpha]$). Our main goal in this thesis was to find constructions for 3-PSTS ($[n \cdot \alpha, n \cdot \alpha, n \cdot \alpha]$) when the degree of each point is as large as possible. The hope is that we can then generalize techniques given in [2] to find 3-PSTS ($[n \cdot \alpha, n \cdot \beta, n \cdot \gamma]$). One such technique is what we call "reduction". For example, the 3-PSTS ($[6 \cdot 2, 6 \cdot 3, 6 \cdot 4]$) can be obtained by deleting 2 sets of N_{120} triples, 1 set of N_{012} triples and 2 sets of N_{201} triples from our construction of a 3- PSTS ($[6 \cdot 8, 6 \cdot 8, 6 \cdot 8]$).

We also plan to augment the Transfer lemma [10], the Equivalence lemma and the Reduction lemma in [2] which we hope will also help in our quest of the ultimate goal of finding necessary and sufficient conditions for the existence of 3-PSTS ($[n \cdot \alpha, n \cdot \beta, n \cdot \gamma]$).

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